Introductory lecture notes on boundary element method

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Chapter 1

First things first

Most engineering problems involve solution of a given differential (or integral) equation with appropriate boundary conditions. The real physical system, through some assumptions is converted to a physical model, which then is formulated into a mathematical model that has to be solved (Fig-1.1). The solution of the mathematical system would yield to the determination of the act of the physical system. But, in most cases, a closed-form mathematical solution is impossible, which in turn requires a numerical solution.
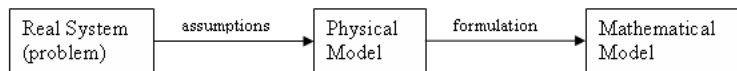

Figure 1.1: From Real System to Mathematical Model

The Boundary Element Methods (BEM) is a numerical method to solve Boundary Value Problems (BVP). In BVP, the governing equations (GE) which are given by the mathematical model in the volume V (See Fig-1.2) are to be solved subject to some boundary conditions (BC) on the boundary S (it is a general joke to refer to this figure as the continuum potato, since it looks like a little potato and represents a continuous media).
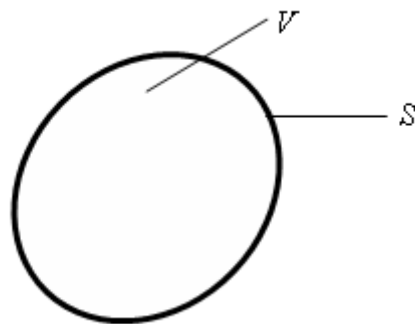

Figure 1.2: The solution region (the continuum potato)

Since it is not possible, in most cases, to obtain a closed-form solution to the problem, numerical methods are inevitable. Among these numerical methods, aside from BEM, most popular ones are; Finite Element Method (FEM), Finite Difference Method (FDM), Finite Volume Method (FVM), Boundary Node Method (BNM) and other Meshless Methods (MM).

Later, we will discuss the advantages and disadvantages of BEM over the other methods. For now, we will focus on the preliminaries that will help us make the BEM formulation. For this, we will first focus on the simplest boundary value problem: the Laplace equation.

1.1 Laplace Equation

The simplest form for the Laplace equation is given as

$$k\nabla^2 u = 0 \tag{1.1}$$

within a domain V having a boundary S. In Equation 1.1, k represents a corresponding material property for the solution domain (can be, for example, the heat conduction coefficient for heat transfer equations) and $u(x, y, z)$ is the field variable that is to be found. The Laplace operator, ∇^2 is defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.2}$$

The problems with this equation appear in many areas of engineering and they are generally called as the potential problems. This is because, a function ϕ is said to be potential if it satisfies

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1.3}$$

Some examples of such engineering problems that can be said to be potential problems are: the steady-state heat transfer, acoustics, flow of ideal fluids, electrochemical reactions, static electric field. In each problem stated, the field variable u (or ϕ) defines a different quantity to be found (e.g., for heat transfer problems it is the temperature difference from the reference temperature) and k represents a material property that is related with the material of the solution domain.

1.1.1 A Note on the notation

The definition of the Laplace operator stems from the Laplace equation for steady-state heat transfer. The heat flux at any point $\mathbf{P}$ in, for example, $x-$direction is defined as

$$q_x = k\frac{\partial u}{\partial x} \tag{1.4}$$

It is very practical to use x_1 for $x-$coordinate, x_2 for $y-$coordinate and x_3 for $z-$coordinate. With this new notation, Equation 1.4 can be re-written as

$$q_1 = k\frac{\partial u}{\partial x_1}$$

which can be generalized for three orthogonal coordinates as

$$q_i = k\frac{\partial u}{\partial x_i} \tag{1.5}$$

for $i = 1, 2, 3$ (representing $x-$, $y-$ and $z-$coordinates). We call this new notation where we name the orhogonal Cartesian coordinates by giving them numbers as the *indicial notation*. Laplace's equation for heat flux is given as

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} = 0 \tag{1.6}$$

This equation states, for no internal heat generation, the net flux on the surfaces of a differential volume should be zero. Note that, one can write this in a simpler form as

$$\sum_{n=1}^{3} \frac{\partial q_n}{\partial x_n} = 0 \tag{1.7}$$

If, instead of q_n in Equation 1.7, we write the expression given in Equation 1.5, we would obtain

$$\sum_{n=1}^{3} \frac{\partial}{\partial x_n}\left(k\frac{\partial u}{\partial x_n}\right) = 0 \tag{1.8}$$

Assuming k to be constant, we would obtain

$$k \sum_{n=1}^{3} \frac{\partial^2 u}{\partial x_n^2} = k \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) = 0 \tag{1.9}$$

Note that, if we redefine the Laplace operator which is given in Equation 1.2 with our new coordinate system notation (e.g.. x_i-notation) as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{1.10}$$

we end up with the form in Equation 1.1.

It is also a good mathematical practice to note that the gradient operator being

$$\nabla \equiv \frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k} \tag{1.11}$$

with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ being the unit normal vectors along x_1, x_2 and x_3 directions respectively, the Laplace operator is given as

$$\nabla^2 = \nabla \cdot \nabla \tag{1.12}$$

where $\cdot$ represents the dot product. Thus, it can be said that, with u being a scalar field, the Laplace operator arises when we take the divergence of the gradient of that scalar field.

1.2 Definitions for the solution domain

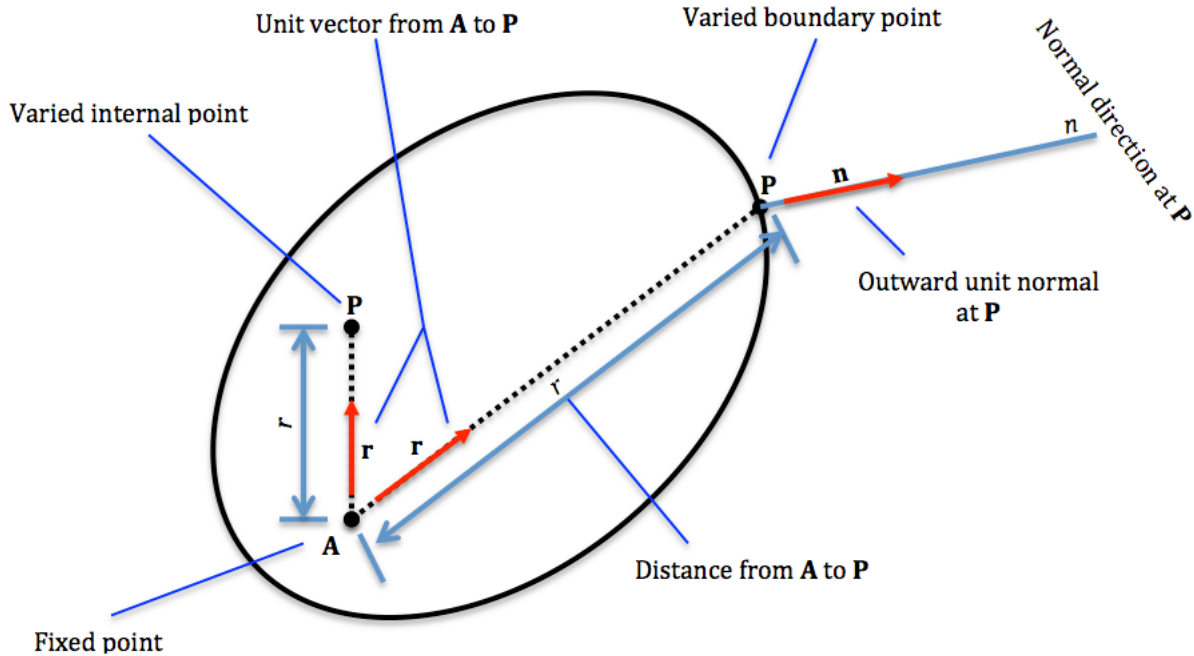At this point, it would be better to introduce some definitions. For this, please refer to Fig-1.3

Figure 1.3: The continuum potato revisited for definitions

We restate that the domain of the solution region (inside the potato) is referred to as V and the boundary of this solution region is denoted by S. We select a point within the solution domain that we will base our calculations (it is something like an observation point that we fix in the solution domain). We call this point as the Fixed point (FP) and in the formulations denote it as $\mathbf{A}$. We also define another point, $\mathbf{P}$, that can change the location within the solution region (and on the boundary).

We will call this point as the varied internal point (when inside the domain) or varied boundary point (when on the boundary of the domain), or in both cases, the varied point (VP). The Euclidean distance between the VP and the FP will be denoted by r, and the unit vector along the $\vec{\mathbf{AP}}$ direction is denoted by $\mathbf{r}$ (the components of $\mathbf{r}$ are denoted by r_i). When the VP is on the boundary, there is an outward normal to the boundary at the VP, which will be denoted by n and the unit normal vector along this direction will be denoted by $\mathbf{n}$ (the components of $\mathbf{n}$ are denoted by n_i).

It is a common practice to refer to the coordinates of the FP as a_i and the coordinates of the VP as x_i. For simplicity in later formulations, we will denote the relative position of the VP with respect to the FP as $\mathbf{Y}$ with components being

$$y_i = x_i - a_i \tag{1.13}$$

It is easy to show that

$$
\begin{aligned}
r^2 &= (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 \\
&= \sum_{n=1}^{3} (x_n - a_n)(x_n - a_n) \\
&= \sum_{n=1}^{3} y_n y_n \tag{1.14}
\end{aligned}
$$

With this, one can obtain the distance as

$$r = \sqrt{\sum_{n=1}^{3} y_n y_n} \tag{1.15}$$

Also, one can note that, the components of $\mathbf{r}$ are given as

$$r_i = \frac{y_i}{r} = \frac{(x_i - a_i)}{\sqrt{\sum_{n=1}^{3} y_n y_n}} \tag{1.16}$$

1.2.1 A short practice

Let us find the gradient of the distance between the FP and the VP. That is, we are in search of ∇r. Note that the gradient of a scalar function is a vector which is defined as

$$\nabla r = \frac{\partial r}{\partial x_j} \mathbf{i} + \frac{\partial r}{\partial x_2} \mathbf{j} + \frac{\partial r}{\partial x_3} \mathbf{k} \tag{1.17}$$

or in a more simplified manner, the components of ∇r are given as $\dfrac{\partial r}{\partial x_i}$. Note that, if we call $s = r^2$, then $\nabla r = \nabla(s^{1/2})$. To find ∇r, we proceed as follows: first, we apply the differentiation rule:

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} s^{-\frac{1}{2}} \frac{\partial s}{\partial x_i} \tag{1.18}$$

It is easy to state that $s^{-\frac{1}{2}} = \frac{1}{r}$. To find $\partial s / \partial x_i$:

$$\frac{\partial s}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\sum_{n=1}^{3} y_n y_n \right) = \sum_{n=1}^{3} \frac{\partial y_n}{\partial x_i} y_n + \sum_{n=1}^{3} y_n \frac{\partial y_n}{\partial x_i} = 2 \sum_{n=1}^{3} \frac{\partial y_n}{\partial x_i} y_n \tag{1.19}$$

but noting that $\partial y_n / \partial x_i = 1$ if $n = i$ and 0 if $n \neq i$, this becomes (in open form)

$$
\begin{aligned}
\frac{\partial s}{\partial x_1} &= 2 \sum_{n=1}^{3} \frac{\partial y_n}{\partial x_1} y_n \\
&= 2 \left(1 \times y_1 + 0 \times y_2 + 0 \times y_3 \right) \\
&= 2 y_1
\end{aligned}
$$

similarly

$$\frac{\partial s}{\partial x_2} = 2y_2$$
$$\frac{\partial s}{\partial x_3} = 2y_3$$

therefore we can say

$$\frac{\partial s}{\partial x_i} = 2y_i \tag{1.20}$$

Inserting this into Equation 1.18 with the definition of $r_i = y_i/r$ given in Equation 1.16 we get

$$\nabla r \equiv \frac{\partial r}{\partial x_i} = r_i \tag{1.21}$$

Thus, it can be said that, the gradient of the distance function gives the components of the unit vector along the measured distance direction.

1.2.2 The directional derivative

For later use in mathematical formulations, we need to define the directional derivative (DD) of a function. Assume that you have a scalar function ϕ defined over a volume V. It is desired to find the change of this function along a given direction, say h. Assume that $\mathbf{h}$ is the unit vector along the $h-$direction. Assume a point $\mathbf{Q}$ that is Δh away from the point $\mathbf{P}$.
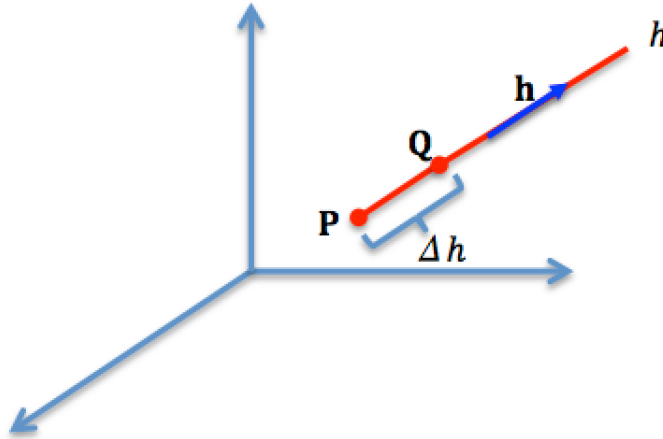

Figure 1.4: The geometical representation of the DD

The DD is defined as

$$\frac{\partial \phi}{\partial h}(\mathbf{P}) = \lim_{\Delta h \to 0} \frac{\phi(\mathbf{Q}) - \phi(\mathbf{P})}{\Delta h} \tag{1.22}$$

Geometrically, it may be shown that

$$\frac{\partial \phi}{\partial h}(\mathbf{P}) = \mathbf{h} \cdot \nabla \phi(\mathbf{P}) \tag{1.23}$$

1.2.3 A short discussion on the DD of the distance function between two points

Assume that we define the distance function $r(\mathbf{A}, \mathbf{P})$ as the distance between the fixed point $\mathbf{A}$ and the varied point $\mathbf{P}$. Recall equations (1.13) and (1.16). From Equation 1.23 we have, for DD of r,

$$\frac{\partial r}{\partial h}(\mathbf{P}) = \mathbf{h} \cdot \nabla r$$

but we have proven befor (in section 1.2.1 that $\nabla r = \mathbf{r}$. Therefore

$$\frac{\partial r}{\partial h}(\mathbf{P}) = \mathbf{h} \cdot \mathbf{r} = \sum_{i=1}^{3} h_i r_i = h_1 r_1 + h_2 r_2 + h_3 r_3 \tag{1.24}$$

In the context of boundary elements, generally, the directional derivative is used to calculate the DD of the distance function defined at a boundary point in the direction of the outward unit normal at that boundary point. Thus, the outward unit normal at the boundary denoted by $\mathbf{n}$, this DD can be expressed as

$$\frac{\partial r}{\partial n}(\mathbf{P}) = \mathbf{n} \cdot \mathbf{r} = \sum_{i=1}^{3} n_i r_i = n_1 r_1 + n_2 r_2 + n_3 r_3 \tag{1.25}$$

1.3 Kronecker's Delta and the summation convention

1.3.1 Kronecker's Delta

At this point, it is better to define the Kronecker's delta, δ_{ij}:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.26}$$

Note that, previously, in the short practice we have said: $\partial y_n / \partial x_i = 1$ if $n = i$ and 0 if $n \neq i$. This is to say

$$\frac{\partial y_n}{\partial x_i} = \delta_{ni} \tag{1.27}$$

It also needs emphasizing a property of Kronecker's delta that we have seen in the short practice:

$$\sum_{n=1}^{N} \delta_{ni} v_n = v_i \tag{1.28}$$

independently from the range of n .

1.3.2 Summation Convention

Also, at this point, it would be very advantageous to define the summation convention. Note that, we used a summation sign in our formulations. Einstein, in his formulations, used a different notation that drops the summation sign. Please skim over the formulations where you see a summation sign:

- Eq(1.7) : $\sum_{n=1}^{3} \frac{\partial q_n}{\partial x_n} = 0$: In this equation, the summation is done over the index n, which is repeated in differentiation

- Eq(1.8) : $\sum_{n=1}^{3} \frac{\partial}{\partial x_n} \left(k \frac{\partial u}{\partial x_n} \right)$: In this equation, again, the summation is done over the index n, which is repeated in differentiation

- Eq(1.15) : $r = \sqrt{\sum_{n=1}^{3} y_n y_n}$: In this equation, again, the summation is done over the index n, which is repeated in multiplication

- Eq(1.28) : $\sum_{n=1}^{N} \delta_{ni} v_n = v_i$: In this equation, again, the summation is done over the index n, which is repeated in multiplication

- Eq(1.25) : $\sum_{i=1}^{3} n_i r_i$: In this equation, the summation is done over the index i, which is repeated in multiplicatiron

We define the summation convention as such: A repeated index in a term in the equation implies summation over the range of that index.

With this definition, it becomes possible to drop the summation term in our equations. For example,

$$\sum_{n=1}^{N} \delta_{ni} v_n = v_i \rightarrow \text{Instead of this form we write this equation as } \rightarrow \delta_{ni} v_n = v_i$$

or,

$$r = \sqrt{\sum_{n=1}^{3} y_n y_n} \rightarrow \text{Instead of this form we write this equation as } \rightarrow r = \sqrt{y_n y_n}$$

It is important to note that, there is no summation over the index n in equations like

$$y_n + x_n = p_n$$

In this case, the index n repeats itself in different terms of the equation. But, if the equation was like

$$p = x_n y_n$$

then, it would imply

$$p = \sum_{n=1}^{N} x_n y_n$$

1.3.3 Definition of the free index and the dummy index

If an index in a term is repeated only once, it is called a free index. If it is repeated twice (that it implies summation) then it is called a dummy index. An index is not allowed to appear more than twice.

A property of the dummy index is: to change the letter for the index does not affect the equation. That is to say, in an equation like

$$K_{ij} = \delta_{ij} \varepsilon_{kk} + 3\varepsilon_{ij}$$

the index k implies summation over its range as:

$$K_{ij} = \delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk} + 3\varepsilon_{ij}$$

assuming the range of k is from 1 to 3. To write this equation with the index k or another letter, like m does not change the equation:

$$K_{ij} = \delta_{ij} \varepsilon_{mm} + 3\varepsilon_{ij}$$

gives the same equation as before. But, if you have changed the letter representation for a free index, you have to make the same change in all terms of the equations. For example, if you want to change the letter i in the term K_{ij} in above equations with the letter n, it is not only to make this change in K_{ij} only:

$$K_{in} = \delta_{ij} \varepsilon_{mm} + 3\varepsilon_{ij}$$

is meaningless. You should change the equation as:

$$K_{in} = \delta_{in} \varepsilon_{mm} + 3\varepsilon_{in}$$

Please practice over other equations that has summation sign to obtain them in a form that has no summation.
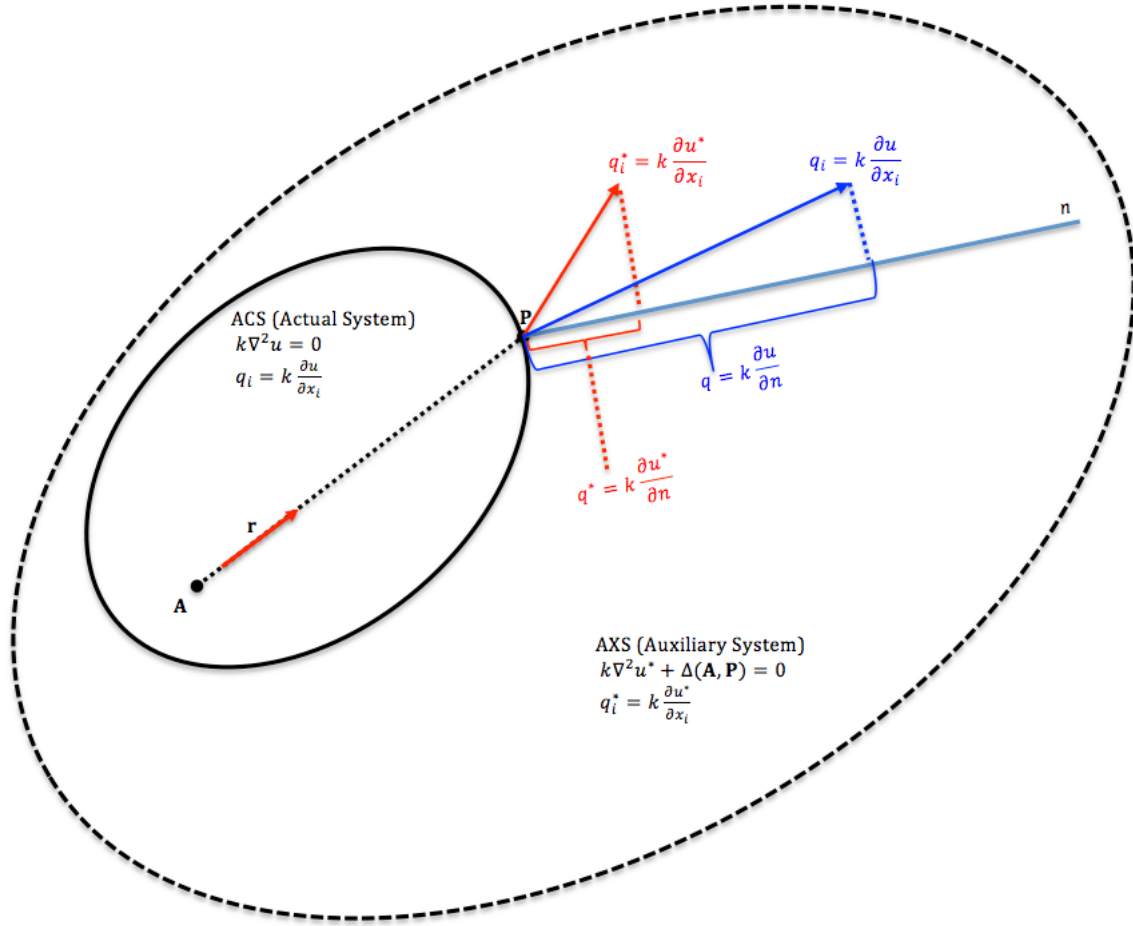
Figure 1.5: Representation of ACS and AXS and some more definitions

1.4 Boundary Element Formulation of the Laplace Equation

Up to now, we have defined the problem, and its solution domain. We call this system as the actual system (will be denoted by ACS from now on). To proceed with the boundary element formulation, we introduce an auxiliary system (will be denoted by AXS) that has the same material properties and that encloses the ACS. In most formulations, this AXS is selected as the infinite space (Fig-1.5).

In the ACS, the governing equations are given by Equation 1.1) and internal flux, q_i, is given by Equation 1.5. In a similar manner, we define a new quantity u^* in the AXS and the flux associated with this new quantity is given by

$$q_i^* = k\frac{\partial u^*}{\partial x_i} \tag{1.29}$$

The governing equations in the AXS are assumed to be

$$k\nabla^2 u^* + \Delta(\mathbf{A}, \mathbf{P}) = 0 \tag{1.30}$$

At this point, we define the Dirac Delta Function:

1.4.1 The Dirac Delta Function and its properties

Dirac Delta Function is the mathematical representation for a unit point source, where its intensity is infinite but the integration over the domain is unity:

$$\begin{aligned} \Delta(\mathbf{A}, \mathbf{P}) = 0 \quad &\text{if} \quad \mathbf{A} \neq \mathbf{P} \\ \Delta(\mathbf{A}, \mathbf{P}) \to \infty \quad &\text{if} \quad \mathbf{P} \to \mathbf{A} \end{aligned} \tag{1.31}$$

or in integral form

$$\begin{array}{ll} \int_V \Delta(\mathbf{A}, \mathbf{P}) \mathrm{d}V = 0 & \text{if} \quad \mathbf{A} \notin V \\ \int_V \Delta(\mathbf{A}, \mathbf{P}) \mathrm{d}V = 1 & \text{if} \quad \mathbf{A} \in V \end{array} \qquad (1.32)$$
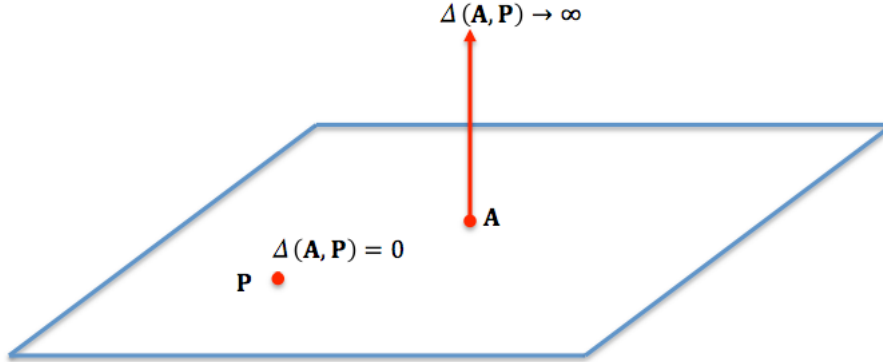

Figure 1.6: Dirac Delta function (point representation)

What is this function? As stated before, it is the mathematical representation for a unit source at the point $\mathbf{A}$. You can see the singularity of this function in Equation 1.31: The value of the function is zero for every point $\mathbf{P}$, which is different than the application point of the source, $\mathbf{A}$. If the point $\mathbf{P}$ is at the application point, than the value of the function jumps to infinity (Fig-1.6). The power of the source is equal to unity, and this is seen in equation Equation 1.32. That means, if the point $\mathbf{A}$ is outside the control volume, the total of the source is zero (since if this is the case, that means there is no internal source), and if it is in the volume, this total source sums up (integral) to unity (although the intensity of the function is infinite, the integral is unity). Here, we impose a special property of the Dirac Delta Function (DDF): Any function $f(\mathbf{P})$, when multiplied by the DDF at a point $\mathbf{A}$, the integral is

$$\begin{array}{ll} \int_V f(\mathbf{P}) \Delta(\mathbf{A}, \mathbf{P}) \mathrm{d}V = 0 & \text{if} \quad \mathbf{A} \notin V \\ \int_V f(\mathbf{P}) \Delta(\mathbf{A}, \mathbf{P}) \mathrm{d}V = f(\mathbf{A}) & \text{if} \quad \mathbf{A} \in V \end{array} \qquad (1.33)$$

We will use this property in a short while. A 1D representation of the DDF is given in Fig-1.7.

1.4.2 Betti's reciprocal theorem for Laplace Equation

Now we have two different solution domains (ACS and AXS) with two different field variables (u and u^* respectively) and two different derived variables (q_i and q_i^* respectively). These derived variables are related to the field variables through equations (1.5) and (1.29). The governing equations are given by the equations (1.1) and (1.30) respectively. The formulation that we will introduce now, leads to the direct formulation in boundary element method (BEM): In the direct formulation, the equations are derived using physical quantities (like temperature, flux, etc.). There is an indirect formulation, too, that leads to the same set of equations, but the derivation is not performed using physical quantities, instead mathematically meaningful parameters are used, which are not necessarily point to a physical quantity.

We start our formulation by proving the so-called Betti's reciprocal theorem for Laplace equations. For this, we first define the product:

$$\frac{\partial u}{\partial x_i} \cdot q_i^* = \frac{\partial u}{\partial x_i} \cdot \left(k \frac{\partial u^*}{\partial x_i} \right) \qquad (1.34)$$

Note that there is a summation over the index i. Since k can change the location in the equation,

$$\frac{\partial u}{\partial x_i} \cdot \left(k \frac{\partial u^*}{\partial x_i} \right) = \left(k \frac{\partial u}{\partial x_i} \right) \frac{\partial u^*}{\partial x_i} = q_i \cdot \frac{\partial u^*}{\partial x_i}$$
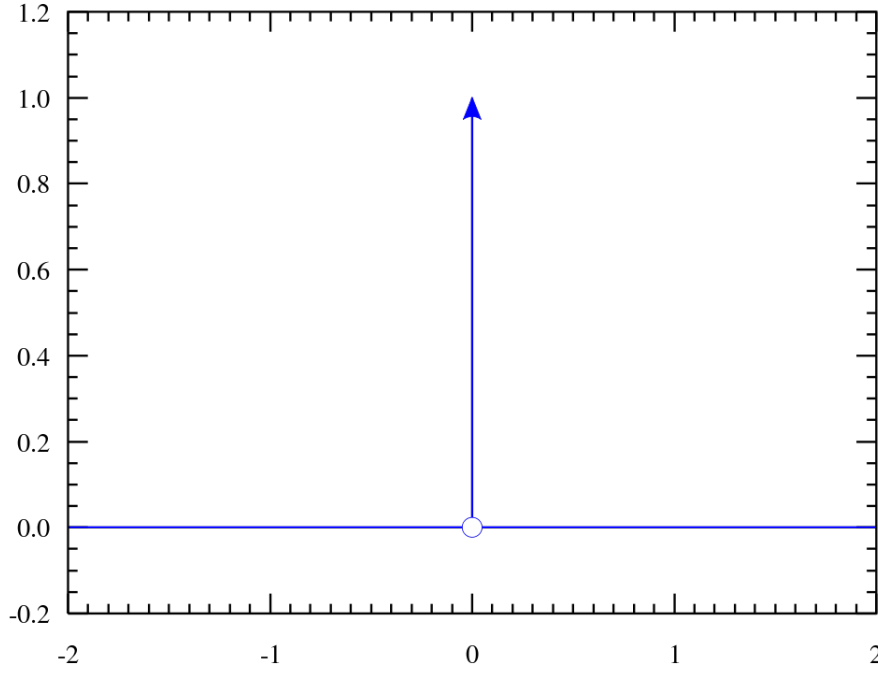
Figure 1.7: 1D representation for Dirac Delta function (copied from Wikipedia)

this implies

$$\frac{\partial u}{\partial x_i} \cdot q_i^* = q_i \cdot \frac{\partial u^*}{\partial x_i} \tag{1.35}$$

This equation is called as Betti's reciprocal theorem (BRT) and forms the starting point for our BEM formulations.

1.4.3   The Reciprocity equation for Laplace Equation

In BRT, we can take the volume integral of the both sides

$$\int\limits_V \frac{\partial u}{\partial x_i} \cdot q_i^* \mathrm{d}V = \int\limits_V q_i \cdot \frac{\partial u^*}{\partial x_i} \mathrm{d}V \tag{1.36}$$

Note that, chain rule requires

$$\frac{\partial}{\partial x_i} \left(u \cdot q_i^* \right) = \frac{\partial u}{\partial x_i} \cdot q_i^* + u \cdot \left(\frac{\partial q_i^*}{\partial x_i} \right)$$

which can be re-arranged as

$$\frac{\partial u}{\partial x_i} \cdot q_i^* = \frac{\partial}{\partial x_i} \left(u \cdot q_i^* \right) - u \cdot \left(\frac{\partial q_i^*}{\partial x_i} \right) \tag{1.37}$$

a similar relation can be written for the right hand side (RHS) of the equation as

$$\frac{\partial u^*}{\partial x_i} = \frac{\partial}{\partial x_i} \left(u^* q_i \right) - u^* \left(\frac{\partial q_i}{\partial x_i} \right) \tag{1.38}$$

Inserting equations (1.37) and (1.38) into (1.36) we get

$$\int\limits_V \frac{\partial}{\partial x_i} \left(u \cdot q_i^* \right) \mathrm{d}V - \int\limits_V u \cdot \left(\frac{\partial q_i^*}{\partial x_i} \right) \mathrm{d}V = \int\limits_V \frac{\partial}{\partial x_i} \left(u^* q_i \right) \mathrm{d}V - \int\limits_V u^* \left(\frac{\partial q_i}{\partial x_i} \right) \mathrm{d}V \tag{1.39}$$

It can be said that, with equations (1.1) and (1.5)

$$\left(\frac{\partial q_i}{\partial x_i}\right) = 0$$

and similarly, with equations (1.29) and (1.30)

$$\left(\frac{\partial q_i^*}{\partial x_i}\right) + \Delta(\mathbf{A}, \mathbf{P}) = 0$$

With these relations, Equation 1.39 can be re-written as

$$\int_V \frac{\partial}{\partial x_i}\left(u \cdot q_i^*\right) \mathrm{d}V + \int_V u \cdot \Delta(\mathbf{A}, \mathbf{P})\mathrm{d}V = \int_V \frac{\partial}{\partial x_i}\left(u^* q_i\right) \mathrm{d}V \tag{1.40}$$

Since the DDF is given with relation to two points, namely the fixed point $\mathbf{A}$ and the varied (integration) point $\mathbf{P}$, it would be better to re-write this equation considering these points:

$$\int_V \frac{\partial}{\partial x_i}\left(u(\mathbf{P}) \cdot q_i^*(\mathbf{A}, \mathbf{P})\right) \mathrm{d}V + \int_V u(\mathbf{P}) \cdot \Delta(\mathbf{A}, \mathbf{P})\mathrm{d}V = \int_V \frac{\partial}{\partial x_i}\left(u^*(\mathbf{A}, \mathbf{P}) \cdot q_i(\mathbf{P})\right) \mathrm{d}V \tag{1.41}$$

Note that, since the governing equation of the AXS is related to the DDF which is a function of both of the points, $\mathbf{A}$ and $\mathbf{P}$, the solution, u^* and the flux q_i^* associated with the solution is a function of the point $\mathbf{A}$ and $\mathbf{P}$. The field variable of the ACS, u, is a function of position, therefore in the integrals we associate it with the position of the point $\mathbf{P}$. The same property applies to the associated flux, q_i.

At this point, we can recall the integral property of the DDF, given in Equation 1.33 for the second integral on the left hand side (LHS) of the equation:

$$\int_V u(\mathbf{P}) \cdot \Delta(\mathbf{A}, \mathbf{P})\mathrm{d}V = u(\mathbf{A}) \tag{1.42}$$

Inserting this into Equation 1.41 we obtain

$$u(\mathbf{A}) + \int_V \frac{\partial}{\partial x_i}\left(u(\mathbf{P}) \cdot q_i^*(\mathbf{A}, \mathbf{P})\right) \mathrm{d}V = \int_V \frac{\partial}{\partial x_i}\left(u^*(\mathbf{A}, \mathbf{P}) \cdot q_i(\mathbf{P})\right) \mathrm{d}V \tag{1.43}$$

1.4.4   The Gauss Integral Theorem

It is time to get rid of the volume integrals in Equation 1.43. For this, we first recall the basic knowledge on gauss integral theorem: For a function (scalar or vector function) defined over the domain V bounded by the surface S, for any operator that we will denote with $\square$, we have the equality

$$\int_V \nabla \square \phi \cdot \mathrm{d}V = \int_S \mathbf{n} \square \phi \cdot \mathrm{d}A \tag{1.44}$$

This is the most general form of the Gauss integral theorem (GIT). In the given equation $\mathbf{n}$ represents the outward unit normal vector at the corresponding poin on the enclosing surface. We will later apply this theorem to other type of problems. At this point, we will use the operator for dot product:

$$\int_V \nabla \cdot \phi \cdot \mathrm{d}V = \int_S \mathbf{n} \cdot \phi \cdot \mathrm{d}A \tag{1.45}$$

Now let us try to convert this to the form in equation (1.43). For this, first we write Equation 1.45 in indicial form as

$$\int_V \frac{\partial \phi_i}{\partial x_i}\mathrm{d}V = \int_S n_i \phi_i \mathrm{d}A$$

If we denote

$$\phi_i = u \cdot q_i^*$$

then it is easy to show

$$\int_V \frac{\partial}{\partial x_i} \left(u(\mathbf{P}) \cdot q_i^*(\mathbf{A}, \mathbf{P}) \right) \mathrm{d}V = \int_S n_i \left(u(\mathbf{P}) \cdot q_i^*(\mathbf{A}, \mathbf{P}) \right) \mathrm{d}A \tag{1.46}$$

similarly, if we denote

$$\phi_i = u^* \cdot q_i$$

then this leads to

$$\int_V \frac{\partial}{\partial x_i} \left(u^*(\mathbf{A}, \mathbf{P}) \cdot q_i(\mathbf{P}) \right) \mathrm{d}V = \int_S n_i \left(u^*(\mathbf{A}, \mathbf{P}) \cdot q(\mathbf{P}) \right) \mathrm{d}SA \tag{1.47}$$

1.4.5 The Boundary element equation (BEE)

Inserting (1.46) and (1.47) into (1.43) we obtain

$$u(\mathbf{A}) + \int_S n_i \left(u(\mathbf{P}) \cdot q_i^*(\mathbf{A}, \mathbf{P}) \right) \mathrm{d}A = \int_S n_i \left(u^*(\mathbf{A}, \mathbf{P}) \cdot q(\mathbf{P}) \right) \mathrm{d}A$$

but note that

$$n_i q_i = n_i k \frac{\partial u}{\partial x_i} = k \frac{\partial u}{\partial n} = q$$

and

$$n_i q_i^* = q^* \tag{1.48}$$

thus the equation turns to

$$u(\mathbf{A}) + \int_S q_i^*(\mathbf{A}, \mathbf{P}) \cdot u(\mathbf{A}) \cdot \mathrm{d}A = \int_S u_i^*(\mathbf{A}, \mathbf{P}) \cdot q(\mathbf{P}) \cdot \mathrm{d}A \tag{1.49}$$

This equation is the so-called Boundary Element Equation (BEE). As can be seen from Equation 1.49 all integrals are on the boundary of the region. If one can find a way to evaluate the surface integrals in this equation, it is possible to find the solution $u(\mathbf{A})$ at any point within the solution domain. Since it is almost impossible to find a closed-form solution to these integrals even with very simplified geometries and boundary conditions, it is our intention in this lecture to attempt a numerical solution to the problem.

For simplicity, we will firstly work on 2D applications of the Laplace Equation. In 2D, a similar analysis can be followed, the only difference being the domain is two dimensional enclosed by a curve in 2D (which is in fact a 1D structure). A similar analysis in 2D would reveal to the same BEE, the only difference being that instead of surface integrals, we are left with line integrals:

$$u(\mathbf{A}) + \int_C q_i^*(\mathbf{A}, \mathbf{P}) \cdot u(\mathbf{A}) \cdot \mathrm{d}S = \int_C u_i^*(\mathbf{A}, \mathbf{P}) \cdot q(\mathbf{P}) \cdot \mathrm{d}S \tag{1.50}$$

1.5 Fundamental solutions for the Laplace Equation

Previously, we have constructed an AXS for formulating the BEE. In this AXS, we have the field variable as u_i^* and the derived variable as q_i^*. The governing equations for the AXS were given as in Equation 1.30 and the flux is defined as in Equation 1.29. At this point, we attempt the solution of these equations in infinite domain (these are therefore called as the Kelvin Solutions or the Green's functions in infinite domain). First of all, we have to note that, since the infinite medium extends

to infinity in all directions, there will be spherical-symmetry (the solution won't change in terms of direction - it will change with distance only). When we use the spherical coordinates, the Laplace operator reads

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Box}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Box}{\partial \theta}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial^2 \Box}{\partial \psi^2} \tag{1.51}$$

If we consider that there is no dependence on θ and ψ, this equation reduces to

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Box}{\partial r}\right) \tag{1.52}$$

Thus, the Equation 1.30 becomes

$$k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u^*}{\partial r}\right) + \Delta(\mathbf{A}, \mathbf{P}) = 0 \tag{1.53}$$

The solution of this equation requires the partitioning of the domain: the infinitessimal domain that is in close neighborhood of the point $\mathbf{A}$, and the rest of the domain. In the rest of the domain, since the domain does not contain the point $\mathbf{A}$, the solution is the solution of the homogenous equation

$$k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u^*}{\partial r}\right) = 0$$

which is simply

$$u^*(\mathbf{A}, \mathbf{P}) = c_1 + \frac{c_2}{r} \tag{1.54}$$

The constant c_1 is the reference solution - the value of the solution which is constant over all the domain. Therefore, it can be omitted (can be assumed to be zero, assuming a zero-reference). To find c_2, a small sphere of radius ϵ is observed around the point $\mathbf{A}$ and the limit as $\epsilon \to 0$ is considered. With some mathematical manipulations, it becomes

$$c_2 = -\frac{1}{4\pi k}$$

for which, u^*, which is mostly named as the first fundamental solution is obtained as

$$u^*(\mathbf{A}, \mathbf{P}) = -\frac{1}{4\pi k \cdot r(\mathbf{A}, \mathbf{P})} \tag{1.55}$$

Here, $r(\mathbf{A}, \mathbf{P})$ refers to the Euclidean distance between the points $\mathbf{A}$ and $\mathbf{P}$. With (1.29) and (1.48) we can obtain the flux solution, q^*, which is mostly named as the second fundamental solution

$$q^*(\mathbf{A}, \mathbf{P}) = -\frac{1}{4\pi r^2}\frac{\partial r}{\partial n} \tag{1.56}$$

A similar solution can be done for 2D analysis, where in this case, there is a circular symmetry (the solution is independent of the angle, depending only on distance). In this case, we can use the polar coordinates for which the laplace operator is defined as

$$\nabla^2 = \frac{\partial^2 \Box}{\partial r^2} + \frac{1}{r}\frac{\partial \Box}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Box}{\partial \theta^2} \tag{1.57}$$

which, for circular symmetry requires the equation

$$k\left(\frac{\partial^2 u^*}{\partial r^2} + \frac{1}{r}\frac{\partial u^*}{\partial r}\right) + \Delta(\mathbf{A}, \mathbf{P}) = 0 \tag{1.58}$$

with the solution

$$u^*(\mathbf{A}, \mathbf{P}) = -\frac{1}{2\pi k}\ln(r) \tag{1.59}$$

leading to

$$q^*(\mathbf{A}, \mathbf{P}) = -\frac{1}{2\pi r}\frac{\partial r}{\partial n} \tag{1.60}$$

1.6   The Reduced boundary element equation (RBEE)

Eq(1.49), in 3D, or Equation 1.50, in 2D, are given for the case where $\mathbf{A}$ is in the solution domain - therefore the integral $\int_V u(\mathbf{P}) \cdot \Delta(\mathbf{A}, \mathbf{P})dV$ gives $u(\mathbf{A})$ as in Equation 1.42. It is easy to show that, this integral yields to zero when the fixed point $\mathbf{A}$ is outside the solution domain. The main problem arises when the point $\mathbf{A}$ is on the boundary - because, it is neither totally in the solution domain, nor totally out of it.

Here, we will not go into details of the derivation; just present the logic (in 2D). Assume that the point $\mathbf{A}$ lies on a smooth surface. Let us construct a circle around the point $\mathbf{A}$ that has a radius ϵ (see Fig-1.8).
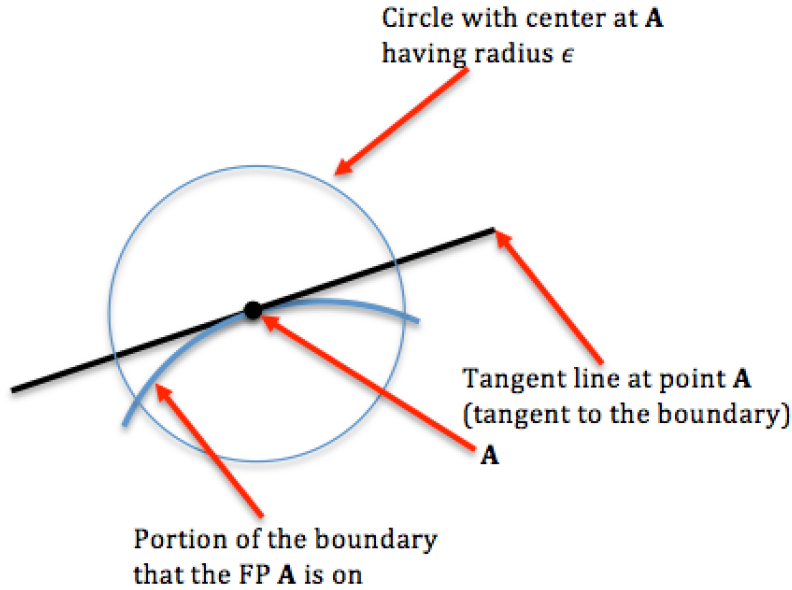

Figure 1.8: A 2D representation of the smooth boundary case

Since the boundary is smooth at the point $\mathbf{A}$, there is one and only one tangent line that can be drawn on the boundary at the point $\mathbf{A}$ and that tangent line divides the circle into two equal half-circles (having the same area). Even if we decrease the radius of the circle, this fact won't change: this is a mathematical way of saying that half of the point $\mathbf{A}$ is in the solution domain and the other half of it is outside. Arising from this fact, it can be shown that, for the fixed point $\mathbf{A}$ being on a smooth portion of the boundary,

$$\int_V u(\mathbf{P}) \cdot \Delta(\mathbf{A}, \mathbf{P})dV = \frac{1}{2}u(\mathbf{A}) \tag{1.61}$$

This is also valid for 3D analysis (due to the same reason, only change being in this case we employ a sphere instead of a circle around the point $\mathbf{A}$).

In case of $\mathbf{A}$ being on an edge (Fig-1.9) it can be seen that, the internal angle being α, only the $\alpha/2\pi$ portion of the circle can be considered to be in the solution domain. Note that α is in radians. Therefore, again, it can be shown that

$$\int_V u(\mathbf{P}) \cdot \Delta(\mathbf{A}, \mathbf{P})dV = \frac{\alpha}{2\pi}u(\mathbf{A}) \tag{1.62}$$

From Equation 1.62 it can easily be seen that for $\alpha = \pi$, which represents smooth surface, the constant on the right hand side becomes $1/2$ just as Equation 1.61 requires.

A smilar relation can be given for 3D applications, where the circle is replaced by sphere and the lines that construct the edge are replaced by the surfaces that coincide at the edge. With above information,
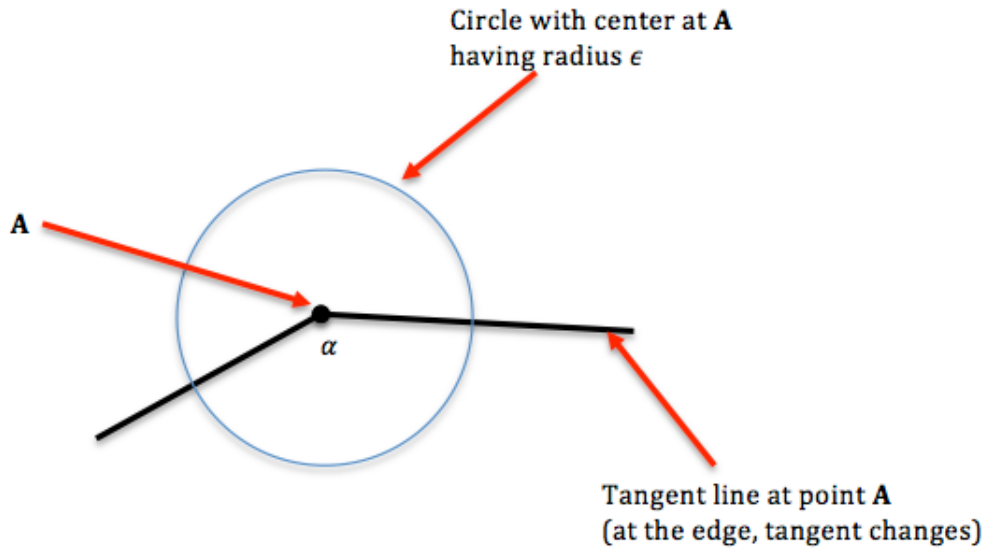
Figure 1.9: A 2D representation of boundary edge

eg., equations (1.61), (1.62) and (1.42) we can re-write equations (1.49) and (1.50) as

$$c(\mathbf{A})u(\mathbf{A}) + \int_S q_i^*(\mathbf{A}, \mathbf{P}) \cdot u(\mathbf{A}) \cdot \mathrm{d}A = \int_S u_i^*(\mathbf{A}, \mathbf{P}) \cdot q(\mathbf{P}) \cdot \mathrm{d}A \qquad (1.63)$$

and

$$c(\mathbf{A})u(\mathbf{A}) + \int_C q_i^*(\mathbf{A}, \mathbf{P}) \cdot u(\mathbf{A}) \cdot \mathrm{d}S = \int_C u_i^*(\mathbf{A}, \mathbf{P}) \cdot q(\mathbf{P}) \cdot \mathrm{d}S \qquad (1.64)$$

with the definition of $c(\mathbf{A})$ ad

$$
\begin{aligned}
c(\mathbf{A}) \quad &= 1 \quad &&\text{if} && \mathbf{A} \in V \\
&= 0 \quad &&\text{if} && \mathbf{A} \notin V \\
&= {}^1\!/_2 \quad &&\text{if} && \mathbf{A} \text{ is on a smooth boundary} \\
&= {}^\alpha\!/_{2\pi} \quad &&\text{if} && \mathbf{A} \text{ is on an edge with angle } \alpha
\end{aligned}
\qquad (1.65)
$$

Eq(1.63) and (1.64) are called the reduced boundary element equations (RBEE). From this point on, we're left with the solution of this RBEE, which is the topic for the next chapter.

Chapter 2

Numerical Solution

As stated before, it is almost impossible to obtain a closed-form (analytical) solution to the BEE that is given in Equation 1.63 or Equation 1.64 when the geometry of the domain and/or the imposed boundary conditions are icomplex. Therefore, a numerical solution is required. This chapter is totally devoted to such numerical solutions. For simplicity, we will work on 2D applications. Later, we will extend our knowledge on 3D applications too.

Note that, what we intend to do basically is to find a proper way to evaluate the boundary integrals (surface integrals in 3D and line integrals in 2D). For this, let us first introduce a proper way to numerically evaluate any bounded integral, such as

$$I = \int_a^b f(x)\mathrm{d}x \tag{2.1}$$

There are lots of methods to numerically evaluate such integrals. But we will focus on mainly to the method called the Gauss Quadrature (GQ). The method, in some books, is called as the Gauss-Legendre quadrature, since it depends mainly on the finding roots for Legendre polynomials (but that's not our problem).

To start with, we will first apply a change of variables: assume we have defined a new variable, τ, such as

$$x = \frac{b+a}{2} + \frac{b-a}{2}\tau \tag{2.2}$$

This requires

$$\mathrm{d}x = \frac{b-a}{2}\mathrm{d}\tau \tag{2.3}$$

Note that, for $\tau = -1 \rightarrow x = a$, for $\tau = +1 \rightarrow x = b$, or vice versa. It can be easily stated that

$$I = \int_a^b f(x)\mathrm{d}x = \int_{-1}^{+1} f(x(\tau))\frac{b-a}{2}\mathrm{d}\tau = \int_{-1}^{+1} F(\tau) \cdot J \cdot \mathrm{d}\tau \tag{2.4}$$

where, J is called the Jacobian of the transformation. In our transformation, which is a linear transformation (since x is a linear function of τ), $J = \dfrac{b-a}{2}$. The function $F(\tau)$ is obtained by inserting Equation 2.2 into every location of x in $f(x)$.

Gauss-Legendre Quadrature, or with a more known name, Gauss Quadrature (GQ) approximates this integral as

$$I = \int_{-1}^{+1} F(\tau) \cdot J \cdot d\tau = J \times \sum_{n=1}^{N} F(T_n) \cdot w_n \tag{2.5}$$

In this equation, T_n are the Gauss points (GP) and w_n are the weights (GW) and the order of the GQ is given by the number of the terms in the series expansion, N. Recall, from mathematics, a GQ of order N can exactly evaluate a polynomial of order $2 \times N - 1$. In Section-2.1 we demonstrate this fact.

2.1 A practice on Gauss Quadrature

We will try to approximate the integral

$$I = \int_0^4 (x^3 + 5) \mathrm{d}x = 84$$

For this, we will use a second order GQ, eg. $N = 2$, which would approximate $2 \times 2 - 1 = 3^{rd}$ order polynomial exactly (what we expect now is the numerical solution would be exactly equal to 84). It is easy to find a second order GQ from literature as tabulated data:

n	Exact T_n	Exact w_n	Computer undestands T_n as	Computer understands w_n as
1	$\sqrt{1/3}$	1	$0.5773502691896258 \times 10^0$	$0.100000000000000 \times 10^1$
2	$-\sqrt{1/3}$	1	$-0.5773502691896258 \times 10^0$	$0.100000000000000 \times 10^1$

First, let us evaluate the Jacobian:

$$J = \frac{4 - 0}{2} = 2$$

Note that, the transformation is

$$x = \frac{4 + 0}{2} + \frac{4 - 0}{2}\tau = 2 + 2\tau$$

With this,

$$F(\tau) = (2 + 2\tau)^3 + 5$$

Therefore, GQ witn $N = 2$ can be constructed as

$$
\begin{aligned}
I &= 2 \times \left\{ \left[(2 + 2 \times 0.5773502691896258)^3 + 5 \right] \times 1 + \left[(2 + 2 \times (-0.5773502691896258))^3 + 5 \right] \times 1 \right\} \\
&= 2 \times (36.39600717839002 + 5.603992821609979) \\
&= 84.0000000000000
\end{aligned}
$$

You can see that the answer matches the exact result. Of course, this would not be possible if the function was not a polynomial.

You may find a small program code written in SciLab the directory:

`/code/scilab/ch2/Gauss_Integrate.sce`

In the code, a 10-pt Gauss Quadrature is used. You can change the function in the code to evaluate different functions. Also you can change the integration limits. Doing so, you may complete the table below:

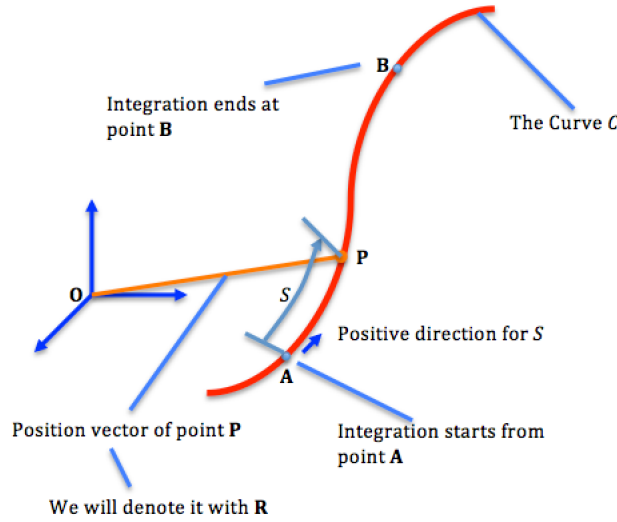| Integral (with limits)	Exact value	Approximate value (code)
$\int_{-2}^{+3} \left(x^6 - 2x^5 + 7 \right) \mathrm{d}x$	144.04761904761904	
$\int_0^{+3} 2\ln(x + 1)\mathrm{d}x$	5.090354888959125	
$\int_0^{+3} \ln(x)\mathrm{d}x$	0.2958368660043291	
$\int_0^\pi \left(\sin(x) \times \cos(2x) \right) \mathrm{d}x$	$-\frac{2}{3} = -0.6666666666666667$	
$\int_0^\pi \left(\sin(3x) \cdot \cos(2x) + 2x^3 + 3x\sin(x) \right) \mathrm{d}x$	59.32932347777059	
$\int_0^1 \frac{1}{x + 1}\mathrm{d}x$	$\ln(2) = 0.6931471805599453$	
$\int_0^1 \frac{1}{x + 0.1}\mathrm{d}x$	$\ln(11) = 2.3978952727983707$	

Figure 2.1: The definitions for the line integral

2.2 The line integral

In 2D, the boundary integral reduces to a line integral as in Equation 1.50. Let us first remember what a line integral is. Assume a function f of position, which we denote as $f(\mathbf{P})$. Note that, in two dimensions, $\mathbf{P}$ had the coordinates (x_1, x_2) which we denote in indicial form as x_i. Therefore, we will denote the function as $f(x_i)$. Also assume a curve, C over which the integral will be taken (see Figure 2.1).

The line integral is evaluated over a domain that starts from the point $\mathbf{A}$ end ends at the point $\mathbf{B}$. The length of the curve along $\mathbf{AB}$ direction is measured with the varialbe S. We will denote the position vector of the point $\mathbf{P}$ as $\mathbf{R}$. Note that, as S changes, $\mathbf{R}$ changes, that means, $\mathbf{R}$ is a function of S, eg., $\mathbf{R} \equiv \mathbf{R}(S)$.

Assume that, when we change S with a little amount, say ΔS, we reach to a point $\mathbf{Q}$ (Figure 2.2). The difference of the two position vectors, $\mathbf{R_P}$ (the position vector at the point $\mathbf{P}$) and $\mathbf{R_Q}$ (the position vector at the point $\mathbf{Q}$) can be denoted as $\Delta \mathbf{R}$. Note that, in two dimensions,

$$\Delta \mathbf{R} = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{2.6}$$

In case that the point $\mathbf{Q}$ gets closer and closer to the point $\mathbf{P}$, the term Δ is replaced by d and it can be shown that,

$$\begin{aligned}
\mathbf{Q} \to \mathbf{P} \mapsto \quad & \Delta \to \mathrm{d} \\
& \Delta \mathbf{R} \to \mathrm{d}\mathbf{R} \\
& \Delta S \to \mathrm{d}S \\
& \|\mathrm{d}\mathbf{R}\| \to \mathrm{d}S \\
& \Delta x \to \mathrm{d}x \\
& \Delta y \to \mathrm{d}y
\end{aligned} \tag{2.7}$$

where $\|\mathrm{d}\mathbf{R}\|$ denotes the length of the vector $\mathrm{d}\mathbf{R}$. It can be proved that, $\mathbf{t}$ being the unit tangent at the point $\mathbf{P}$ in the positive direction of S,

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}S} \tag{2.8}$$

Again, from Figure 2.2, in view of conversions given in Equation 2.7, it can be said that

$$\mathrm{d}S = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{(\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2} = \sqrt{\mathrm{d}x_i \mathrm{d}x_i} \tag{2.9}$$
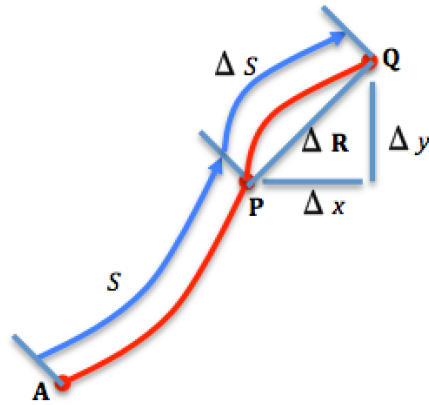
Figure 2.2: More definitions

2.2.1 Definition of the line integral

Remember that we have assumed a function f of the position, $f \equiv f(x_i)$. It is obvious that, for each location, $\mathbf{P}$, on the curve C, this function will have a value given by $f(\mathbf{P})$. Now assume that we subdivide the curve C into N line sections (see Figure 2.3). Each line section is denoted by C_i and has a length denoted by L_i and has a midpoint which is denoted by $\mathbf{P}_i$ (where $i = 1..N$).
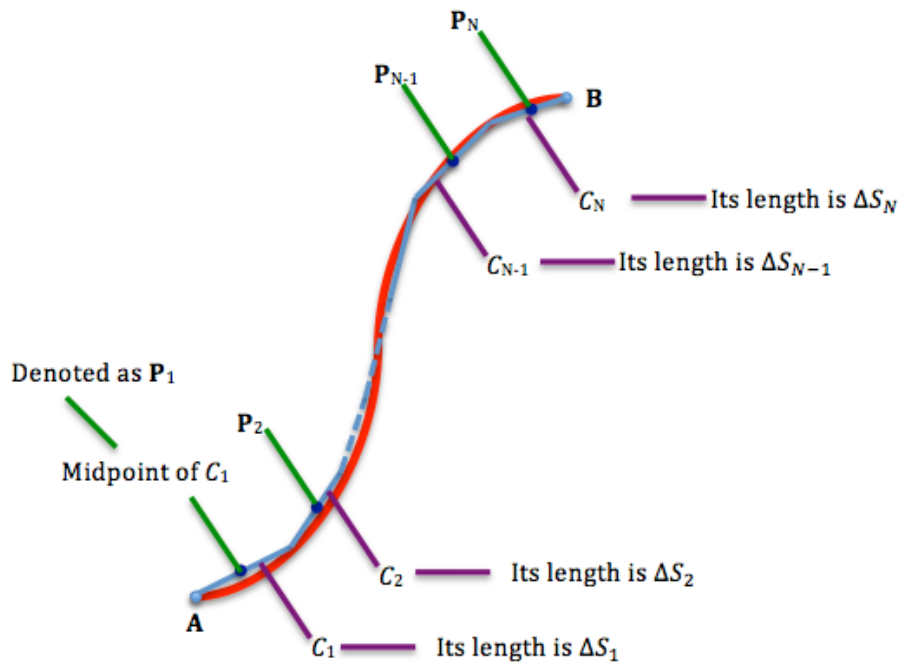

Figure 2.3: The division of the curve into N elements

In this context, we will assume that the variation of the function f over every line segment is constant with the value that it has on the midpoint of the line segment. That is to say, over all the line segment, for example, C_i the value of the function is assumed to be equal to the value $f(\mathbf{P}_i)$. With such an assumption, the term

$$f(\mathbf{P}_i) \times \Delta S_i$$

gives the area given in Figure 2.4, which is an approximate value of the area under the $f - curve$. When we sum all these areas, we would obtain the summation

$$F = \sum_{i=1}^{N} f(\mathbf{P}_i) \cdot \Delta S_i \tag{2.10}$$

where $f(\mathbf{P}_i)$ are the values of the function f evaluated at the center point $\mathbf{P}_i$.
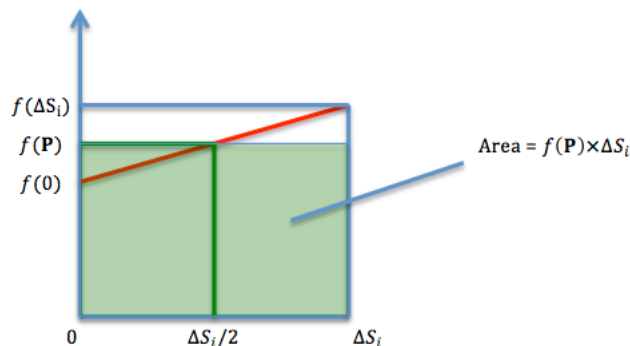

Figure 2.4: The function f over a representative line segment C_i

As you can imagine, we can select N as large as possible, and as $N \rightarrow \infty$, $\Delta S_i \rightarrow \mathrm{d}S$. With this observation, we define the line integral as

$$I = \int_C f(x_i) \cdot \mathrm{d}S = \lim_{N \rightarrow \infty} \sum_{i=1}^{N} f(\mathbf{P}_i) \cdot \Delta S_i \tag{2.11}$$

assuming the given limit converges (exists). At this point, we have to find a way to evaluate I without dividing into infinite number of elements. For this, we express the curve C in a different way:

2.2.2   Parametric representation of a curve

At this point we introduce a parameter, t , to track the points of the curve C in a two-dimensional space which is referred to a Cartesian coordinate system, x_i. That is to say, we try to find a single parameter t and two functions $x_1(t)$ and $x_2(t)$, such that when we change t, the point $\mathbf{P}(x_1(t), x_2(t))$ moves along the curve C (preferably in the positive $S-$direction). A quick example is the parametric representation of the circle:

$$
\begin{aligned}
x_1(t) &= a\cos(t) \\
x_2(t) &= a\sin(t) \\
0 \leq t &\leq 2\pi
\end{aligned}
$$

where a is the radius of the circle (a constant) and t represents the angle from the horizontal axis (mostly, x_1 -axis). You can easily verify that, as we change t from 0 to 2π, the point $\mathbf{P}(a\cos(t), a\sin(t))$ moves along the circle with center at the origin and having a radius of a (see Figure 2.5).

As another simple example, we can give

$$
\begin{aligned}
x_1(t) &= 2\cos(t) + t \\
x_2(t) &= \sqrt{t} \\
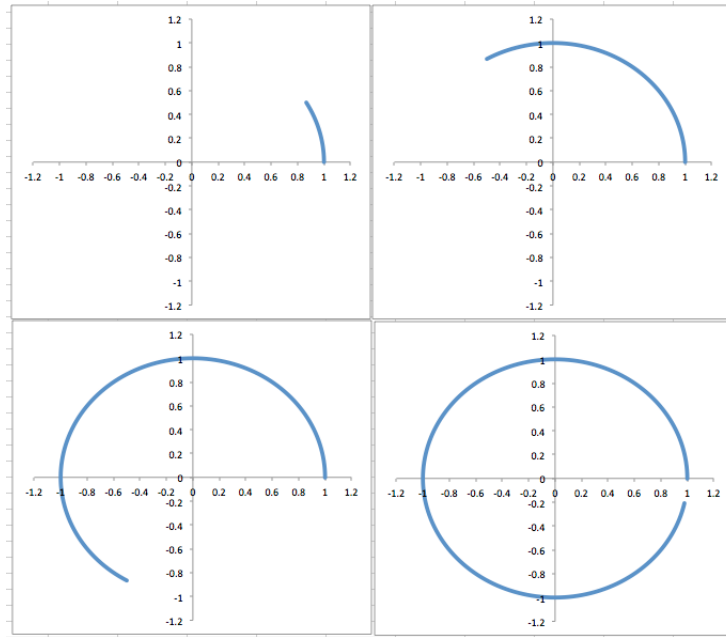0 \leq t &\leq 1
\end{aligned}
$$

Figure 2.5: The evolution of the circle when $a = 1$, graphs are taken at $t = \pi/6$ (top left), $t = 2\pi/3$ (top right), $t = 4\pi/3$ (bottom left) and $t = 355\pi/180$ (bottom right)

this would give the curve in Figure 2.6.

Now, how does this representation of the curve help us? Since we have related the coordinates of any point on the curve to a single parameter t, we can easily relate dx_i to dt. For example, in the case of the circle, we have

$$\begin{aligned} dx_1 &= -a\sin(t) \cdot dt \\ dx_2 &= a\cos(t) \cdot dt \end{aligned}$$

or in the case of the curve in Figure 2.6,

$$\begin{aligned} dx_1 &= (-2\sin(t) + 1)dt \\ dx_2 &= \frac{1}{2\sqrt{t}}dt \end{aligned}$$

Recall Equation 2.9 which states $dS = \sqrt{(dx_1)^2 + (dx_2)^2}$. For the circle, this would imply

$$dS = a \cdot dt$$

and for the curve in Fig-2.6 this would be

$$dS = \sqrt{(-2\sin(t) + 1)^2 + 1/4t} \cdot dt$$

It is also possible to re-write the function $f(x_i)$ with replacing $x_i \to x_i(t)$. Thus the integral becomes

$$I = \int_C f(x_i) \cdot dS = \int_{t_A}^{t_B} f(x_i(t)) \cdot J(t) \cdot dt \tag{2.12}$$

where $J(t)$ is named as the Jacobian. For the circle example, $J(t) = a$, and for the other curve $J(t) = \sqrt{(-2\sin(t) + 1)^2 + 1/4t}$.
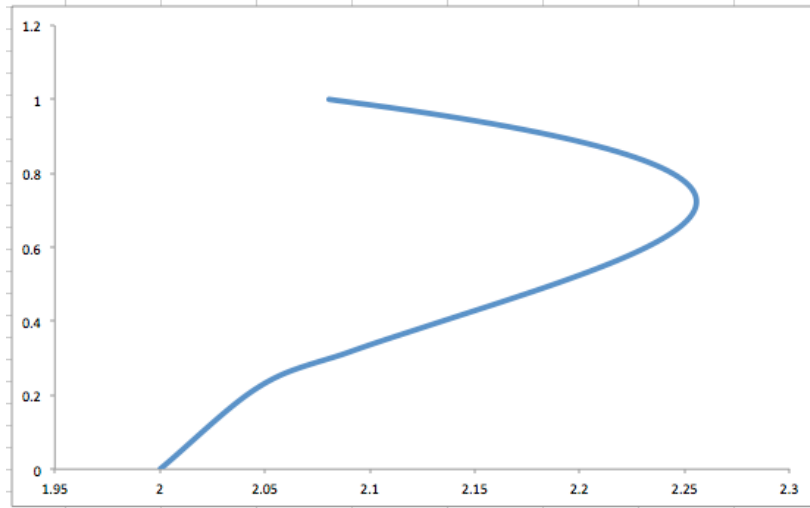
Figure 2.6: A simple curve

2.2.3 A simple example on line integral

Assume that, we want to find the line integral of the function

$$f(x_i) = 2x_1 + x_2^2 \tag{2.13}$$

on the circle with parametric equation

$$
\begin{aligned}
x_1(t) &= 2\cos(t) \\
x_2(t) &= 2\sin(t) \\
0 \leq t &\leq 2\pi
\end{aligned}
\tag{2.14}
$$

With this parametric equation, $f(x_i)$ can be re-written as

$$f(t) = f(x_i(t)) = 4 \times \cos(t) + 4 \times \sin^2(t)$$

Recall that,

$$
\begin{aligned}
\mathrm{d}x_1 &= -2\sin(t)\mathrm{d}t \\
\mathrm{d}x_2 &= 2\cos(t)\mathrm{d}t
\end{aligned}
$$

which leads

$$\mathrm{d}S = \sqrt{4\sin^2(t) + 4\cos^2(t)}\mathrm{d}t = 2\mathrm{d}t$$

Therefore, the integral becomes

$$
\begin{aligned}
I &= \int_C (2x_1 + x_2^2) \cdot \mathrm{d}S \\
&= \int_0^{2\pi} (4\cos(t) + 4\sin^2(t)) \cdot 2 \cdot \mathrm{d}t \\
&= 8 \times \left[\frac{t}{2} + \sin(t) - \frac{1}{4}\sin(2t) \right]_0^{2\pi} \\
I &= 8 \times \pi
\end{aligned}
\tag{2.15}
$$

Here, we should note that, with $f(x_i) = 1$, one obtains the length of the curve (this can be easily verified since each line segment length is summed up). With above circle, to find the circumference of

the circle one can use

$$
\begin{aligned}
I_C &= \int_C (1) \cdot \mathrm{d}S \\
&= \int_0^{2\pi} (1) \cdot 2 \cdot \mathrm{d}t \\
&= 2 \times [t]_0^{2\pi} \\
I_C &= 4\pi
\end{aligned}
$$

which is compatible with what we have learned in Calculus: the circumference of a circle with radius r is equal to $2\pi r$. In this case, $r = 2$, therefore the circumference ended up to be $4\pi = 2\pi(2)$.

2.2.4 Numerical evaluation of line integrals

In case of the line integral taken over a circle, the analytical evaluation both possible and simple. In some cases, though, even if the analytical evaluation is possible, it may not be simple. Also, programming an analytical solution that is applicable to all curves is not simple. Therefore, this section is devoted to numerical evaluation of line integrals. There are of course many ways to evaluate line integrals numerically, but we will employ the simplest way: use the definition of the line integrals.

Recall that, a line integral is defined using a series expansion with dividing the curve into N line segments (and then increasing N to infinity). We will use the same method here with the difference that we will not take N to infinity, we will be satisfied with N being sufficiently large. For a representative figure of this division, refer to Figure 2.7 where the curve C is divided into N line segments using K points on the curve. Each line segment has a start point and an end point that fixes its orientation in 2-D space. For example, line segment (LS) #1 starts with Point (Pt) #1 and ends with Pt#2. Similarly, LS#3 starts with Pt#3 and ends with Pt#4.
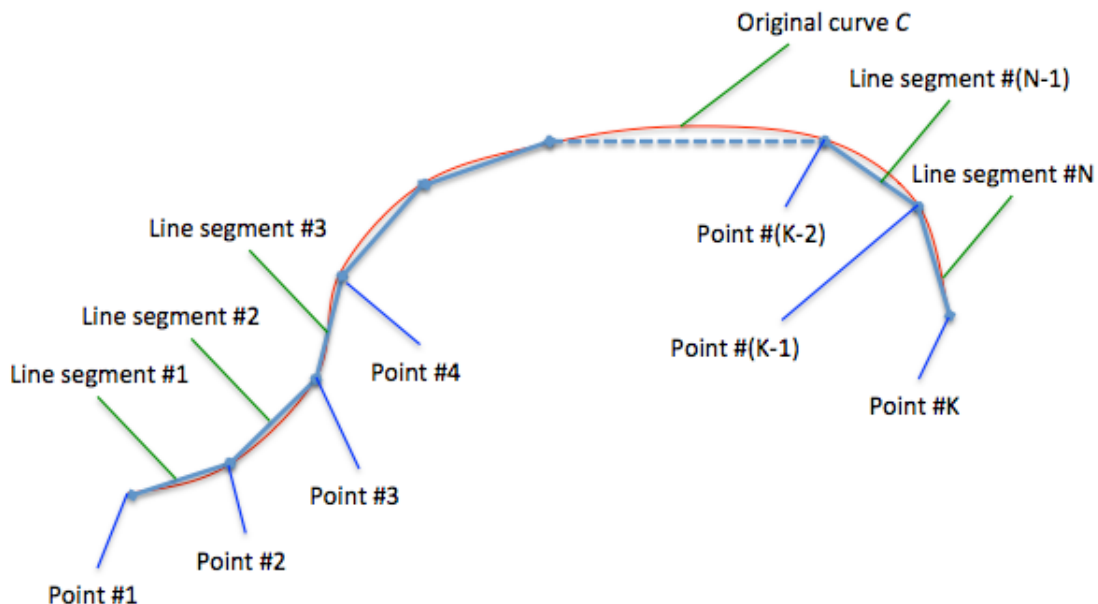

Figure 2.7: A curve divided to line segments

Let us take a representative line segment under magnification, which is presented in Figure 2.8.

There are two different types of numbering for the points on the line segments (as the figures suggest clearly). The first type is called the local numbering: in each LS, the numbering starts with 1 and
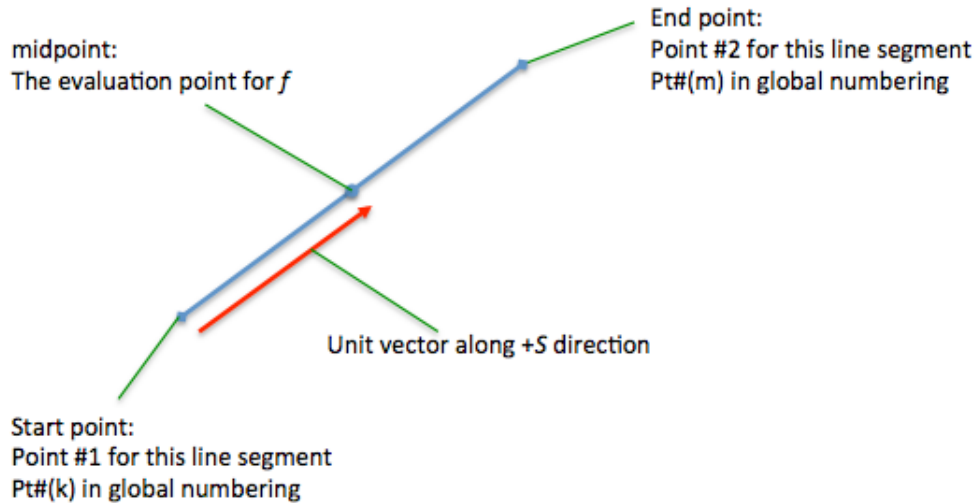
Figure 2.8: A representative line segment under magnification

counts the number of points on the LS. In our case, the number of points is 2, therefore, start point is numbered (locally) as 1 and the end point is numbered as 2. The second type of numbering is called the global numbering. In this case, each point is numbered in a global fashion. The start point can be represented by any number (in Figure 2.8, it is numbered as k), and the end point similarly (numbered as m).

To examplify this, let us take the example of a circle divided into six line segments as in Figure 2.9. The table in the figure presents the global numbering for each local number of each line segment. Such tables (or lists) are named as element connectivity lists, or simply, connectivity. You may find a small program code written in SciLab in:

`/code/scilab/ch2/divide_parametric.sci`

that divides a curve (with a given parametric equation) into N line segments and returns the point coordinates and the connectivity for such a division process. In the code, as a sample parametric equation, a circle is presented; but this can be easily changed by editing the function parametric_equation.

Let us return to our original discussion: we divided the curve into line segments of each we know the orientation by the location of their first and second local points. Since, each division is a line segment, each point on this division should lie on a line - this means that there is a linear corrolation between the coordinates of the points on the line segment and its orientation. There are several different ways (in fact, infinitely many ways) to represent this linear corrolation, one of which is,

$$x_i^{\mathbf{P}} = \frac{x_i^2 + x_i^1}{2} + \frac{x_i^2 - x_i^1}{2}t \qquad (2.16)$$

where $-1 \le t \le +1$. You can verify that this is a linear relation in terms of $x_i^{\mathbf{P}}$ and t . You can also verify that when $t = -1$, we get $x_i^{\mathbf{P}} = x_i^1$, which means the point is located at the first local point of the line segment. Similarly, when $t = +1$, we get $x_i^{\mathbf{P}} = x_i^2$ which means that the point $\mathbf{P}$ is located to the second local point of the line segment. As expected (since this is a linear relation), when $t = 0$, the point $\mathbf{P}$ locates the midpoint of the line segment, therefore

$$x_i^M = x_i^{\mathbf{P}}(t = 0) = \frac{x_i^2 + x_i^1}{2} \qquad (2.17)$$

Well and good, now how does this help us? In fact, above, we have presented a common representation of the parametric equations that can be used for all defined line segments. If we input the coordinates
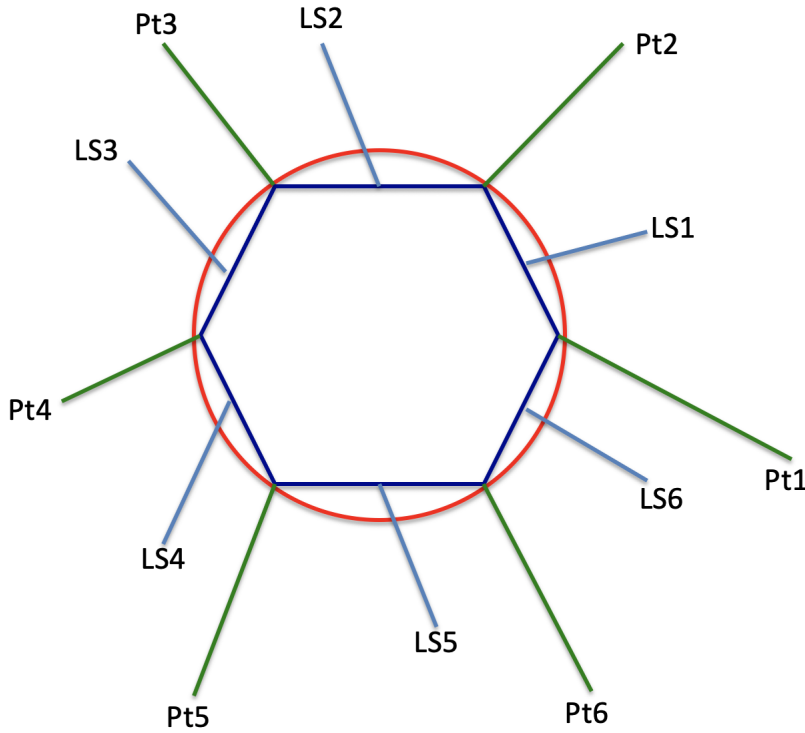
LS #	Local-1	Local-2
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	1

Figure 2.9: Circle divided into six line segments

of the first and second local points of the line segment that we have in hand, we can easily obtain the corresponding parametric equation. We can go further: with Equation 2.16,

$$\mathrm{d}x_i = \frac{x_i^2 - x_i^1}{2}\mathrm{d}t \tag{2.18}$$

In this equation, we dropped the superscript $\mathbf{P}$ for simplicity (and we will do from now on for similar expressions that involve $\mathbf{P}$). Since, eq.(2.9) we have $\mathrm{d}S = \sqrt{\mathrm{d}x_i\mathrm{d}x_i}$, with eq.(2.18) we obtain

$$\begin{aligned}
\mathrm{d}S &= \sqrt{\left(\frac{x_1^2 - x_1^1}{2}\mathrm{d}t\right)^2 + \left(\frac{x_2^2 - x_2^1}{2}\mathrm{d}t\right)^2} \\
&= \sqrt{\frac{\mathrm{d}t^2}{4}\left[\left(x_1^2 - x_1^1\right)^2 + \left(x_2^2 - x_2^1\right)^2\right]} \\
&= \frac{\mathrm{d}t}{2}\sqrt{\left[\left(x_1^2 - x_1^1\right)^2 + \left(x_2^2 - x_2^1\right)^2\right]}
\end{aligned}$$

but note that, the length of the line segment is given as

$$L = \sqrt{\left[\left(x_1^2 - x_1^1\right)^2 + \left(x_2^2 - x_2^1\right)^2\right]} \tag{2.19}$$

Therefore,

$$\mathrm{d}S = \frac{L}{2}\mathrm{d}t \tag{2.20}$$

Any line integral over the nth line segment would therefore become

$$I_n = \int_{C_n} f(x_i) \cdot \mathrm{d}S = \int_{-1}^{+1} f(x_i(t)) \cdot \frac{L}{2} \cdot \mathrm{d}t \tag{2.21}$$

Note that this is a very similar expression to Equation 2.5 with $J = L/2$ and $F(\tau) = f(x_i(t))$. Therefore the integral can be approximated with

$$I_n = \frac{L}{2} \sum_{n=1}^{N} f(x_i(T_n)) \times w_n \qquad (2.22)$$

2.2.5   Wrapping things up

In this section, we will gather up our knowledge on numerical evaluation of line integrals employing a simple example: the line integral over a circle with function given in Equation 2.13. The circle is located at the origin with radius $r = 2$, where the parametric equations are presented in Equation 2.14.

The general block diagram of the way we evaluate the line integral is given in Figure 2.10. First, we divide the curve into N line segments. Over each line segment we use the transformation in Equation 2.16 so that we obtain a parametric relation of the curve. Using this, obtain the function value at this point, where we use GQ to evaluate the line integral corresponding to that line segment, as in Equation 2.22. Then we sum up all the contributions from each line segment, to obtain the final value.
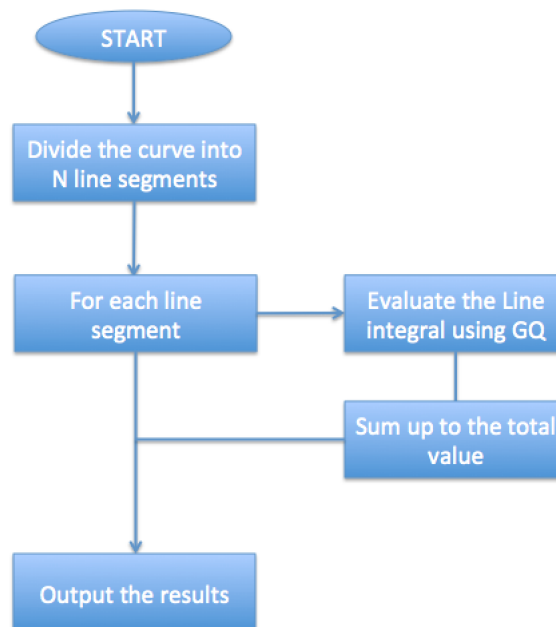

Figure 2.10: Block diagram for line integral evaluation

Please refer to the file ~/code/Ch2/line_integral.cpp to practice this line integral evaluation procedure. You may observe that, with 6 divisions, the line integral value will deviate considerably from the exact result, whereas, as you increase the number of divisions, you get closer to the exact value. The table below shows the integral value obtained from the code with different n values. Compare it with the exact value given in Equation 2.15.

number of line segments (n)	value of the integral from code
6	20.00000
36	24.97374
216	25.12831
1296	25.13262
7776	25.13274
exact	25.13274

Self Practice

Consider a circle with radius $\rho = 2$ units whose center is located at the coordinate-origin. Consider, further, that a fixed point $\mathbf{A}$ is given on this circle (also, for later practice, you may select this fixed point any other point in space) with coordinates $\mathbf{A}(2, 0)$. Modify and use the line integral function to evaluate the line integral

$$\int_C r(\mathbf{A}, \mathbf{P}) \times r(\mathbf{A}, \mathbf{P}) \mathrm{d}S = 32\pi \tag{2.23}$$

where $r(\mathbf{A}, \mathbf{P})$ is the Euclidean distance between the fixed point $\mathbf{A}(2, 0)$ and the varied point (on the circle) $\mathbf{P}(x_1, x_2)$. You can compare the evaluated value with the exact value which is given in Equation 2.23.

2.2.6   Matrix Form of the Boundary Element Equation

Recall that, the RBEE is given as in Equation 1.64. At this stage, we will perform the simplest way to solve this equation. Assume that, we have re-drawn the domain in figure 1.3 using a finite number of line segments as in figure 2.11. To designate the number of line segments, we will denote this quantity by N.
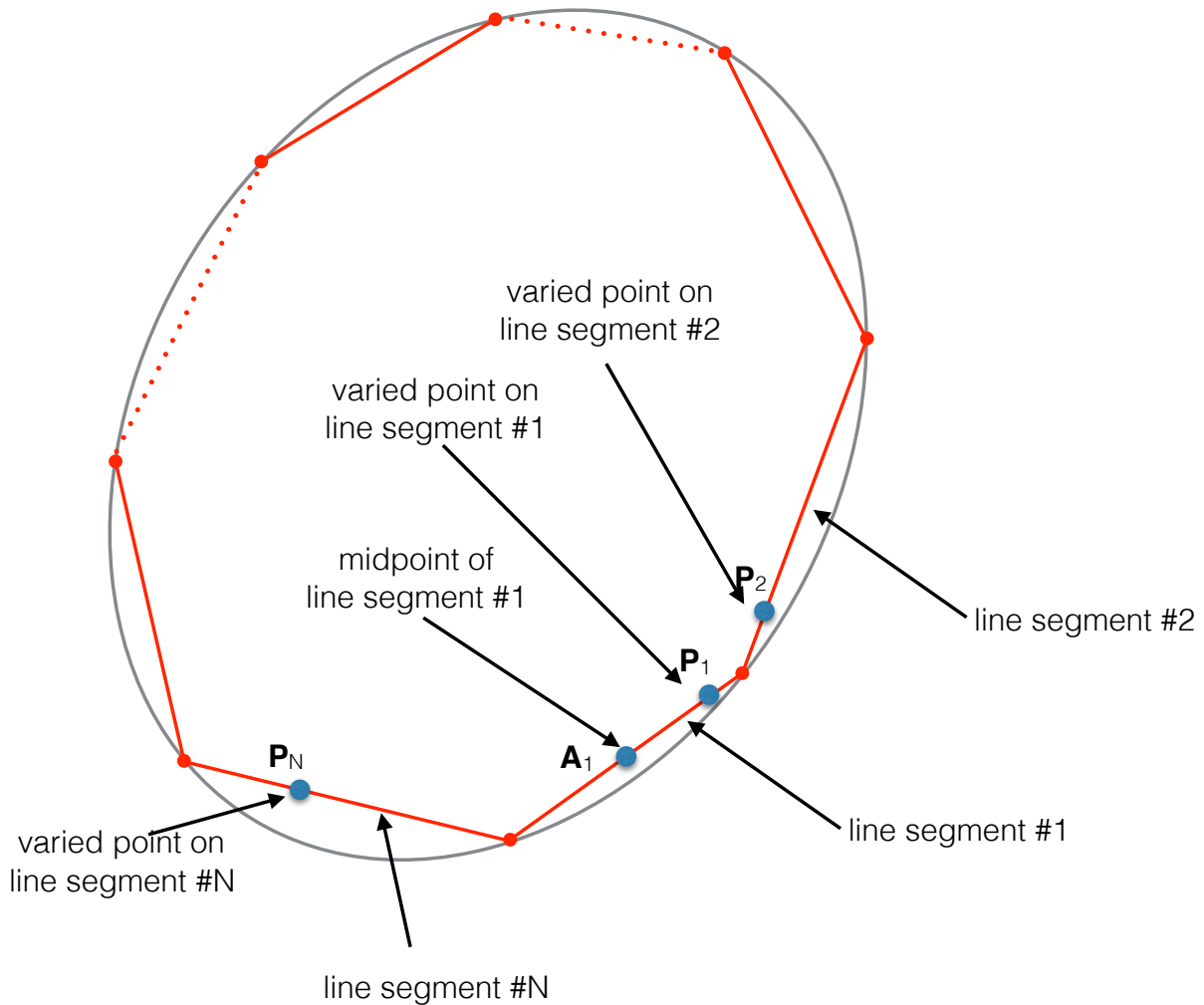

Figure 2.11: Continuum potato divided into line segments

Let us write the equation 1.64 at the midpoint of the first line segment, which will be denoted as $\mathbf{A}_1$, where the subscript 1 will denote that this point belongs to the first line segment. Since the line segment is *smooth* at this location of $\mathbf{A}_1$, the value of $C(\mathbf{A}_1)$ will be $1/2$:

$$\frac{1}{2}u(\mathbf{A}_1) + \int_{C_1} q^*(\mathbf{A}_1, \mathbf{P}_1) \cdot u(\mathbf{P}_1)\mathrm{d}S_1 + \int_{C_2} q^*(\mathbf{A}_1, \mathbf{P}_2) \cdot u(\mathbf{P}_2)\mathrm{d}S_2 + \cdots + \int_{C_N} q^*(\mathbf{A}_1, \mathbf{P}_N) \cdot u(\mathbf{P}_N)\mathrm{d}S_N \quad =$$

$$\int_{C_1} u^*(\mathbf{A}_1, \mathbf{P}_1) \cdot q(\mathbf{P}_1)\mathrm{d}S_1 + \int_{C_2} u^*(\mathbf{A}_1, \mathbf{P}_2) \cdot q(\mathbf{P}_2)\mathrm{d}S_2 + \cdots + \int_{C_N} u^*(\mathbf{A}_1, \mathbf{P}_N) \cdot q(\mathbf{P}_N)\mathrm{d}S_N \quad (2.24)$$

Note that, in equation 2.24, C_i refers to the ith line segment, $\mathbf{P}_i$ refers to the varied point on the line segment C_i, and $\mathrm{d}S_i$ designates that the line integral is taken over the line segment C_i.

At this point, we assume that for all the line segments, the value of u and q are constant with the value evaluated at its midpoint. This assumption, although a very rough approximation, is an assumption that could converge to an exact value as the number of line segments are increased. We denote the midpoint of each line segment with the notation $\mathbf{A}_i$ where the subscript i denotes that it is the midpoint of the line segment C_i. So, mathematically, what we assume can be represented by

$$u(\mathbf{P}_i) = u(\mathbf{A}_i) \quad \forall \mathbf{P}_i \in C_i \tag{2.25}$$

and

$$q(\mathbf{P}_i) = q(\mathbf{A}_i) \quad \forall \mathbf{P}_i \in C_i \tag{2.26}$$

We can further shorten the notation by the definitions

$$u_i = u(\mathbf{A}_i) \quad \text{and} \quad q_i = q(\mathbf{A}_i) \tag{2.27}$$

In view of these definitions and assumptions, equation 2.24 becomes:

$$\frac{1}{2}u_1 + u_1 \cdot \int_{C_1} q^*(\mathbf{A}_1, \mathbf{P}_1) \,\mathrm{d}S_1 + u_2 \cdot \int_{C_2} q^*(\mathbf{A}_1, \mathbf{P}_2) \,\mathrm{d}S_2 + \cdots + u_N \cdot \int_{C_N} q^*(\mathbf{A}_1, \mathbf{P}_N) \,\mathrm{d}S_N \quad =$$

$$q_1 \cdot \int_{C_1} u^*(\mathbf{A}_1, \mathbf{P}_1) \,\mathrm{d}S_1 + q_2 \cdot \int_{C_2} u^*(\mathbf{A}_1, \mathbf{P}_2) \,\mathrm{d}S_2 + \cdots + q_N \cdot \int_{C_N} u^*(\mathbf{A}_1, \mathbf{P}_N) \,\mathrm{d}S_N \qquad (2.28)$$

Note that, in equation 2.28, the integrals

$$H_{1\,j} = \int_{C_j} q^*(\mathbf{A}_1, \mathbf{P}_j) \,\mathrm{d}S_j \tag{2.29}$$

and

$$G_{1\,j} = \int_{C_j} q^*(\mathbf{A}_1, \mathbf{P}_j) \,\mathrm{d}S_j \tag{2.30}$$

can be evaluated using Gauss quadrature.